Quantum scale invariance on the lattice

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Abstract

We propose a scheme leading to a non-perturbative definition of lattice field theories which are scale-invariant on the quantum level. A key idea of the construction is the replacement of the lattice spacing by a propagating dynamical field – the dilaton. We describe how to select non-perturbatively the phenomenologically viable theories where the scale invariance is broken spontaneously. Relation to gravity is also discussed.

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1. Introduction

It is believed that the scale invariance, existing in *classical* field theories without dimensionfull parameters, is generically broken in quantum field theory (for a review see [1]). The known exceptions include finite theories such as $N = 4$ super Yang-Mills [2] or other classes of supersymmetric theories [3], which, however, are lacking an immediate phenomenological relevance. The standard regularisations of the ultraviolet infinities introduce one type of mass scale or another and this scale makes its way into the renormalized theory. The explicit breaking of scale invariance (SI) or its anomalous breaking by quantum effects leads to severe fine-tuning problems, facing the realistic models of particle physics. One of them is the problem of stability of the weak scale against quadratic quantum corrections and the second one is the cosmological constant problem, associated with quartic divergences.

A quantum field theory (gravity included) with *exact*, but spontaneously broken SI would solve the above-mentioned problems. The scale-invariance, existing on quantum level, forbids quadratic or quartic divergences, exactly in a way the gauge invariance keeps the photon massless in the Quantum Electrodynamics. The spontaneous breaking of the dilatational symmetry would introduce all different mass scales, observed in Nature (including the mass of the Higgs boson, the QCD scale Λ and alike, the Newtons gravity constant, etc.) through the vacuum expectation value of the dilaton field. Moreover, a combination of the ideas of SI with those of unimodular gravity [4, 5, 6] leads to a possible explanation of primordial inflation and of late acceleration of the universe [7].

In [8] a perturbative way for construction of a new class of quantum theories, where SI is exact, but broken spontaneously, was suggested. The similar procedure for keeping the local conformal symmetry intact at the quantum level was proposed earlier in [9]. The basic idea is as follows. To have a quantum theory which is scale invariant the renormalisation procedure must not introduce any dimensionfull parameter. This may be achieved, in dimensional regularisation of [10], by

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replacing the 't Hooft-Veltman normalization scale μ by an appropriate combination of dynamical quantum fields in such a way that the SI is preserved in any space-time dimension $d = 4-2\varepsilon$. Clearly, this introduces new interactions when $\varepsilon \neq 0$. Though their strength is suppressed by ε, they leave a trace in the renormalized theory, as their combination with the poles in ε coming from counter-terms leads to finite contributions. The procedure described above is only possible if the SI is spontaneously broken (otherwise the perturbative expansion is ill defined), but this is required anyway by phenomenological considerations.

If the construction of quantum scale-invariant theories based on dimensional regularisation is self-consistent, then the result should have a more general character and other renormalisation schemes, leading to quantum dilatational invariance, should exist. Lattice regularisation plays a special role in construction and studies of quantum field theories since it allows for non-perturbative approach. The aim of the present letter is to attempt a non-perturbative lattice construction of the quantum SI theories.

The paper is organized as follows. In Section 2 we present the idea of lattice regularisation of SI theories with the use of an example of a simple scalar theory. In Section 3 we will discuss the inclusion of gravity. Section 4 is conclusions.

2. Lattice spacing as a dynamical field

To present our main idea, we start with a simple scalar field model, given by the action

$$S = \int d^4x \, \mathscr{L}, \tag{1}$$

$$\mathscr{L} = \frac{1}{2}\left[(\partial_\mu \chi)^2 + (\partial_\mu h)^2\right] - V(h,\chi), \tag{2}$$

$$V(h,\chi) = \lambda \left(h^2 - \zeta^2\chi^2\right)^2 + \beta\chi^4 \, . \tag{3}$$

The fields h and χ can be thought of as the Higgs field and the dilaton correspondingly. Here λ and ζ are dimensionless coupling constants. The theory is scale-invariant at the classical level: the action does not change under substitution $h(x) \to \Omega h(\Omega x)$ and $\chi(x) \to \Omega\chi(\Omega x)$, where Ω is a parameter of dilatation transformation. If $\beta > 0$, the vacuum state of the theory $h = \chi = 0$ is scale-invariant. If $\beta < 0$, the theory does not have a ground state. The vacuum state with spontaneously broken SI appears if $\beta = 0$ and corresponds to some point in the flat direction $h^2 - \zeta^2\chi^2 = 0$. The specific values of coupling constants and of the vacuum expectation value of the dilaton χ_0 are not important for what follows. [1]

The scale-invariant perturbative renormalisation of this model has been considered in [8]. It was shown there that if the parameter μ of dimensional regularisation is replaced by an appropriate combination of dynamical fields, then the counter-terms can be chosen in such a way that the divergences are removed and the classical flat direction, existing at $\beta = 0$, is not lifted by quantum corrections. A choice, motivated by cosmological considerations [7], reads

$$\mu^{2\varepsilon} \to \left[\omega^2\right]^{\frac{\varepsilon}{1-\varepsilon}} \, , \tag{4}$$

where $\omega^2 \equiv \left(\xi_\chi\chi^2 + \xi_h h^2\right)$ and ξ_χ, ξ_h are the couplings of the scalar fields to the Ricci scalar, see section 3. The low energy theory contains a massive Higgs field and the massless dilaton, the latter being a Goldstone boson of the broken scale invariance. [2] A more familiar example of a scale-invariant quantum theory with spontaneous breaking of SI, which appears in standard renormalisation procedure, is $N = 4$ supersymmetric Yang-Mills theory [2].

The requirement that the scale invariance must be spontaneously broken is essential for perturbative construction. Only in this case the perturbative computations can be done. Moreover, if both the ground state and the action respect SI, the resulting theory does not contain any mass scale and thus cannot be accepted phenomenologically.

Let us turn now to lattice regularisation (LR). Consider some space-time lattice of points, with oriented edges connecting them. The space-time is taken to be infinite at the moment. Finite difference scheme can be constructed in many different ways leading to the same result in continuum limit. To simplify the subsequent discussion let us denote by ϕ_{i+} and ϕ_{i-} the values of a generic field ϕ at space-time points corresponding to the i-th edge end. The field difference and the field strength associated to this edge is $\Delta\phi_i = \phi_{i+} - \phi_{i-}$ and $\phi_i = (\phi_{i+} + \phi_{i-})/2$ correspondingly. We assume that the length of the lattice edges a_i may vary in space and time. Then the classical lattice action of the model eq. (3) becomes

$$S = \sum_i a_i^4 \left[\frac{\Delta\chi_i^2 + \Delta h_i^2}{2a_i^2} - V(h_i,\chi_i)\right] \, . \tag{5}$$

[1] The choice required by phenomenological considerations corresponds to $\chi_0 \simeq M_\mathrm{P}$, where M_P is the Planck mass, and $\zeta \sim v/M_\mathrm{P} \sim 10^{-16} \lll 1$, where v is the electroweak scale.

[2] This particle has only derivative couplings to matter and thus escapes experimental bounds, given the small value of ζ.

Clearly, this regularisation breaks explicitly the scale invariance. It introduces the lattice spacing a_i, playing the role of the inverse ultraviolet cutoff in quantum theory. Therefore, the only possibility to construct a scale invariant theory at quantum level in the framework of LR, is to promote the lattice spacing to a dynamical field ϕ,

$$a_i^{-1} = \eta \phi_i \ , \qquad (6)$$

where η is some constant, determining how fine is the grid. A possibility to have an ultraviolet cutoff as a dynamical field in SI theories was discussed previously in [11].

The ϕ could be a new field, not present yet in eq. (3). However, we can assume that our starting action contains already all scalar fields of the theory. Then, in general, ϕ is a function of h and χ. We take it to be $\phi^2 = \omega^2$, in analogy with the previous discussion, see eq. (4). Then the action on the lattice becomes

$$S = \sum_i \left[\frac{\Delta\chi_i^2 + \Delta h_i^2}{2\omega_i^2 \eta^2} - \frac{V(h_i, \chi_i)}{\omega_i^4 \eta^4} \right] . \qquad (7)$$

It is invariant under the scale transformations

$$h_i \to \Omega h_i, \quad \chi_i \to \Omega \chi_i \ . \qquad (8)$$

The quantum SI system can be defined by the Euclidean partition function,

$$Z = \prod_i \int \frac{d\chi_i dh_i}{\sigma_i^2} \, e^{-S_E}, \qquad (9)$$

where we introduced the scale-invariant path integral measure and made the redefinition of the fields (in such a way that $\omega_i^2 \to \sigma_i^2 = h_i^2 + \chi_i^2$) and coupling constants accounting for renormalisation as follows:

$$S_E = \sum_i \left[\frac{A\Delta\chi_i^2 + B\Delta h_i^2}{2\sigma_i^2} + \frac{C\chi_i^4 + D\chi_i^2 h_i^2 + E h_i^4}{\sigma_i^4} \right] , \qquad (10)$$

where A, B, C, D and E are 5 arbitrary parameters. If the theory is considered in a finite space-time volume, the summation in (10) is limited by the total number N of the lattice edges.

The naive continuum limit of the lattice theory (the physical volume is fixed) is achieved by the scaling $\eta \to \kappa\eta$, $(A,B) \to (A,B)/\kappa^2$, $(C,D,E) \to (C,D,E)/\kappa^4$, $N \to \kappa^4 N$, $\kappa \to \infty$, corresponding to a finer covering of the space by the lattice points.

For numerical lattice simulations another choice of variables can be more convenient. For example, the change of variables $\chi = \exp(\theta)\cos(\phi)$, $h = \exp(\theta)\sin(\phi)$ will simplify the kinetic and potential terms.

The phenomenologically interesting theories are those where the scale invariance is spontaneously broken. This does not necessarily happen for all possible choices of parameters in the lattice action (10). To select a class of theories with SI spontaneously broken, one can construct an effective potential $V_{eff}(\chi,h)$, given by

$$\exp\left(-N\sigma^{-4} V_{eff}(\chi,h)\right) = \frac{1}{Z} \prod_i \int \frac{d\chi_i dh_i}{\sigma_i^2} \delta\left(1 - \frac{\bar{\chi}}{\chi}\right) \delta\left(1 - \frac{\bar{h}}{h}\right) e^{-S_E} \ , \qquad (11)$$

where $\sigma^2 = \chi^2 + h^2$ and

$$\bar{\chi} = \frac{1}{N}\sum_i \chi_i, \quad \bar{h} = \frac{1}{N}\sum_i h_i \ . \qquad (12)$$

Note that the factor of $N\sigma^{-4}$ appears in the l.h.s. of eq. (11) because we have to recover integration over classical space-time. Due to SI, the effective potential is $V_{eff}(\chi,h) = \sigma^4 W(x)$, where $x = h/\chi$. To get a spontaneous breaking of SI the parameters of the action must be chosen in such a way that the minimum of $W(x)$ is achieved at some point x_0 such that $\sigma \neq 0$ and

$$\lim_{N\to\infty} W(x_0) = 0 \ . \qquad (13)$$

In terms of field variables the minimum of $W(x)$ correspond to the flat direction, $h = x_0\chi$. This condition singles out some 4 - dimensional surface in 5-dimensional parameter-space, as it happens in the classical theory or in quantum theory with perturbative SI renormalisation prescription.

Several important comments are now in order.

(i) A continuum limit of the lattice model (10) may lead to a non-interacting trivial field theory (as in the case of $\lambda\phi^4$ theory [12, 13, 14]) even on the surface corresponding to spontaneous breaking of scale-invariance. In this case the action (10) cannot be considered as fundamental and would rather correspond to an effective field theory, valid below some energy scale. This effective theory is exactly scale invariant, with SI spontaneously broken. An extreme point of view which does not require an existence of continuum limit is that the space-time is in fact discrete.

(ii) If the parameters of the lattice action are such that eq. (13) does not hold, the ground state of the theory is scale-invariant. The lattice simulations in these case can be used to determine at the non-perturbative level the anomalous dimensions of different composite operators.

(iii) The measure for functional integral cannot be fixed on the grounds of scale-invariance only. The eq. (9) provides just one of the possible examples. The freedom in the choice of the measure and in the expression

of the field ϕ through h and χ corresponds to the freedom of renormalisation prescription in perturbative SI, discussed in [8].

(iv) One can add to the lattice action (5) infinite number of terms, containing higher dimensional operators suppressed by the lattice spacing, for example $\sum_i a_i^6 \chi_i^6$. These will be transformed to $F \sum_i \chi_i^6/\sigma_i^6$ by our procedure. We expect these terms be irrelevant in the continuum limit, since F scales as F/κ^6 whereas the number of space-time points grows as κ^4 only.

(v) By introducing some artificial constant length scale a_0, we can formally rewrite (7) as an action on the lattice with a constant spacing. Then in the limit $a_0 \to 0$ the continuum Lagrangian corresponding to the action eq. (7) is

$$\mathscr{L}_0 = \frac{1}{2}\frac{M_{\rm P}^2}{\omega^2}\left[(\partial_\mu\chi)^2 + (\partial_\mu h)^2\right] - \frac{M_{\rm P}^4}{\omega^4} V(h,\chi) \ , \quad (14)$$

where $M_{\rm P}^2$ should be understood as the vacuum expectation value of ω. The Lagrangian (14) corresponds to a highly non-linear and non-renormalisable theory. It is intriguing that it looks like the field theory part of the scale-invariant Lagrangian of the scalar fields coupled to gravity after transition to the Einstein frame [7]. One could have started with the bare action eq. (14) right away. But then the intriguing relation of this theory to a dynamical discrete space-time would have been lost.

(vi) Clearly, the notion of the classical space and time can appear in our approach only if the scale-invariance is spontaneously broken and the field σ acquires a non-zero expectation value. The arising space-time is flat in the case of a ground state with $\sigma = $ const. One can imagine several possibilities for emerging of gravity. First of all, an effective low energy theory derived from eqs. (9)-(10) may already contain the Einstein or unimodular gravity. Put it in other words, the correlator of some composite spin 2 operators, constructed from the fields h and χ, may have a massless pole corresponding to the graviton. If true, the Hilbert-Einstein action is effective rather than fundamental. In alternative situation one may have to add to the action (10) an extra term describing the gravity in a conventional way, as discussed below.

3. Inclusion of scale-invariant gravity

The lattice spacing, represented by a quantum field, is nothing but a form of space-time quantization, which could lead eventually to quantum gravity. This approach to the lattice formulation of gravity goes back to Regge [15] (for reviews see, e.g. [16, 17, 18]). The basic idea is that a curved space can be approximated by a collection of piecewise flat manifolds or simplices (triangles for the case of 2-dimensional surfaces, tetrahedra and pentatopes for the case of 3 and 4-dimensions respectively). A d-dimensional simplex has $d-1$-dimensional "sides" or "faces". In turn, faces are bounded by a $d-2$-dimensional "hinges". (In the triangulation of a 2-dimensional surface a hinge is a point.) In a piecewise flat manifold the curvature is concentrated on hinges [15], so they play special role in the formalism. This approach avoids the use of coordinates. In it the lengths of 1-dimensional edges of simplices play the role of dynamical variables, exactly what we need for construction of scale-invariant quantum theories. So, we identify the length of every edge of 4d simplex with inverse of our field σ as in (6).

The standard Einstein action in Regge formalism is written as

$$M_{\rm P}^2 \int d^4x\sqrt{g}R \quad\to\quad M_{\rm P}^2 \sum_{hinges\ i} V_i\delta_i, \qquad (15)$$

where V_i is the area of two dimensional hinge and δ_i is the deficit angle there, equal to 2π minus the sum of the dihedral angles between the faces of the simplices meeting at that hinge.

Of course, this expression is not scale-invariant due to the presence of explicit Planck mass. To get the SI action, we replace $M_{\rm P}$ by a dynamical scalar field which can be chosen to be $G\sigma$, where G is some constant. We arrive to the Brans-Dicke like action

$$S_G = G \sum_{hinges\ i} \delta_i V_i \sigma_i^2 \ . \qquad (16)$$

Under the global scale transformations, the deficit angle does not change, while the area scales as $V_i \propto \sigma_i^{-2}$. Continuum limit (volume is fixed) corresponds to $G \to G/\kappa^2$, $\kappa \to \infty$ (number of hinges scales as κ^2).

The sum in eq. (16) can be explicitly evaluated in the Dynamical Triangulation (DT) approach [19, 20, 21] which is a variant of the Regge calculus where all simplexes are equilateral. Also, the separation of metric variables and the scalar field is not clear if one sticks to the original Regge version, but it becomes transparent if one changes to DT where all the geometric degrees of freedom are in the connectivity of the triangulation. Since in an equilateral simplex in d-dimensional space the angle between two faces sharing a common hinge is given by $\arccos(1/d)$, the deficit angle is given by

$$\delta_i = 2\pi - n_i \arccos(1/d) \ , \qquad (17)$$

where n_i denotes the number of simplices incident on an i-th hinge. In d dimensions a hinge has dimensionality

$j = d-2$ and since hinges are also equilateral, their volume is given by

$$V_i = \frac{a^j\sqrt{j+1}}{j!\sqrt{2^j}} \ , \tag{18}$$

which in four dimensions becomes $V_i = \sqrt{3}a^2/4$. By replacing $a_i^{-1} \to \sigma_i$ we obtain for the gravitational action

$$\begin{aligned} S_R &= G \sum_{hinges\ i} V_i \delta_i \sigma_i^2 \\ &= G \sum_{hinges\ i} \frac{\sqrt{3}}{4}\left(2\pi - n_i \arccos\left(\frac{1}{4}\right)\right) \ , \end{aligned} \tag{19}$$

which reduces to $S_R = k_2 N_2 - k_4 N_4$, where N_2 and N_4 are the numbers of hinges and simplexes in a given simplical manifold, while k_2 and k_4 are some bare constants, cf. [22, 23].

Other matter fields can be added to the Regge formalism as well (see, e.g. [16, 17, 18]) and the theory can be made scale-invariant by the replacement (6). As the lengths of the of edges are invariant under general coordinate transformations, the invariant measure can be chosen as in eq. (9) with the appropriate powers of σ_i.

4. Conclusions

In this letter we argued that it is possible to identify an ultraviolet cutoff, needed for regularisation of particle physics models, with a dynamical field – the dilaton. A concrete realisation of this suggestion with the use of lattice field theories is proposed. This construction may lead to existence of a new class of theories, which are perturbatively and non-perturbatively scale invariant at the quantum level. In this type of theories the problem of stability of the Higgs mass hierarchy and of the cosmological constant are solved automatically.

An intriguing feature of this proposal is that it indicates that a solution of two outstanding problems, namely the origin of different mass scales in particle physics (spontaneous breaking of quantum scale invariance) and quantization of space-time (dynamical cutoff), may be intricately related.

Clearly, a lot of work is required to see if this suggestion goes through. The important question is whether the non-perturbative continuum limit of spontaneously broken SI lattice models, described in this paper, leads to non-trivial theories. The lattice simulations with well developed tools and techniques would allow to approach this question.

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